RAYLEIGH WAVES IN AN INHOMOGENEOUS ELASTIC HALF-SPACE OF WAVEGUIDE TYPE

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The problem of natural oscillation of an elastic isotropic half-space in which the transverse velocity achieves a minimum at some depth, and is monotonous in the rest, is considered. It is assumed that the properties of the medium depend continuously on the depth z.

The high-frequency asymptotic behavior of the spectrum of the problem and the dependence of the eigenfunctions on z are studied. It is shown that the nature of the oscillations depends essentially on whether the minimum of the transverse velocity will be greater or less than the Rayleigh velocity on the surface. The method of asymptotic splitting of a system of ordinary differential equations is used to carry out the investigation.

1. Formulation of the problem; dispersion equation. Let us consider the half-space $z \geqslant 0, -\infty < x, y < +\infty$, occupied by an elastic isotropic medium with the Lamé parameters $\lambda(z), \mu(z)$ and the density $\rho(z)$. The displacement vector $\mathbf{u}(x, y, z, t)$ satisfies the system

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu) \operatorname{grad} \operatorname{div} \mathbf{u} - \mu \operatorname{rot} \operatorname{rot} \mathbf{u} + \operatorname{grad} \lambda \operatorname{div} \mathbf{u} + \\ + 2 (\operatorname{grad} \mu, \nabla) \mathbf{u} + [\operatorname{grad} \mu, \operatorname{rot} \mathbf{u}] \qquad (1.1)$$

Let us consider the plane problem, i.e. $\mathbf{u} = \mathbf{u}(x, z, t) = (u_x, 0, u_z)$. We assume the boundary of the half-space to be free of stresses, i.e. at $z = 0$

$$\tau_{zx} \equiv \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = 0, \qquad \tau_{zz} \equiv 2\mu \frac{\partial u_z}{\partial z} + \lambda \operatorname{div} \mathbf{u} = 0 \qquad (1.2)$$

As $z \to \infty$ the displacements decrease in the sense that

$$\int_0^\infty \rho(z) \, |\mathbf{u}(x, z, t)|^2 \, dz < \infty \qquad (1.3)$$

Let us consider particular solutions of (1.1) of a special kind, which have the character of waves travelling along the x-axis

$$\mathbf{u}_\sigma(x, z, t; k) = (G_1(z, k, \sigma) \sin k(x - t\sigma), 0, G_2(z, k, \sigma) \cos k(x - t\sigma)) = \\ = \operatorname{Re} \{ e^{ik(t\sigma - x)} (iG_1, 0, G_2) \} \qquad (1.4)$$

Here k is the wave number, σ the phase velocity. From (1.1) to (1.3) we obtain a Sturm-Liouville type problem on the $z \geq 0$ half-axis for the two-dimensional vector $\mathbf{G} = (G_1, G_2)$:

$$-(AG')' + k^2BG' - k(CG)' + kC^*G' = k^2\sigma^2\rho EG \tag{1.5}$$

$$AG' + kCG = 0 \quad \text{for } z = 0 \tag{1.6}$$

$$\int_0^\infty \rho(z) \, |G(z, k, \sigma)|^2 \, dz < \infty \tag{1.7}$$

Here

$$A = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}, \quad B = \begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -\mu \\ \lambda & 0 \end{pmatrix}$$

$$C^* = \begin{pmatrix} 0 & \lambda \\ -\mu & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \nu \equiv \lambda + 2\mu \quad \left(' \equiv \frac{d}{dz}\right)$$

It is convenient to introduce $\mathbf{Z} = (Z_1, Z_2, Z_3, Z_4) = (G_1, G_2, k^{-1}G_1', k^{-1}G_2')$, then (1.5) becomes

$$\mathbf{Z}' = (kH + K)\mathbf{Z} \tag{1.8}$$

where 4×4 matrices H and K are defined in [1].

Let $\mathbf{Z}^p$ and $\mathbf{Z}^s$ be any two linearly independent solutions of the system (1.8) satisfying the condition (1.7). The solution of the problem (1.5) to (1.7) may be written as follows

$$G = \alpha G^p + \beta G^s \tag{1.9}$$

Hence, α and β are determined from the algebraic homogeneous system

$$D_p(k, \sigma)\alpha + D_s(k, \sigma)\beta = 0, \qquad E_p(k, \sigma)\alpha + E_s(k, \sigma)\beta = 0$$

with the determinant

$$\Delta(k, \sigma) = D_p(k, \sigma)E_s(k, \sigma) - E_p(k, \sigma)D_s(k, \sigma) \tag{1.10}$$

$$D_l(k, \sigma) = k(Z_3^l - Z_2^l)|_{z=0}, \qquad E_l(k, \sigma) = k\left(Z_4^l + \frac{\lambda}{\nu} Z_1^l\right)\Big|_{z=0} \qquad (l = p, s)$$

The solutions of Eq.

$$\Delta(k, \sigma) = 0 \tag{1.11}$$

henceforth designated the dispersion equation, are connected in an evident manner to the eigen numbers of the problem (1.5) to (1.7); the corresponding vectors $\mathbf{G}(z, k, \sigma)$ will be eigenfunctions of the problem.

The asymptotic behavior of the solutions of the dispersion equation as $k \to \infty$ as well as the dependence of the eigenfunctions on z are studied herein.

As is known [2], in the case of a homogeneous medium the system (1.5) is solved in terms of elementary functions, Eq. (1.11) is independent of k, and for $\sigma > 0$, a unique solution of (1.11) exists, $\sigma \equiv v_R$ (the so-called Rayleigh velocity) wherein

$$0 < v_R < v_s, \quad v_s = \sqrt{\mu / \rho}$$

Here v_s is the transverse velocity. The corresponding solution u_σ decreases exponentially as $z \to \infty$.

The case of an inhomogeneous half-space in which $v_s(z) > v_R$ has been considered in [3], where the dispersion equation has been studied in the domain

$$0 < \varepsilon \leqslant \sigma \leqslant \min v_s(z) - \varepsilon.$$

The domain $\sigma > \min v_s(z)$, wherein the case $v_R > \min v_s(z)$ is admitted, is considered in this paper. In this connection, turning points of the system (1.8) should be examined.

2. Asymptotic of solutions of the system (1.8). The characteristic numbers of the matrix H are [1]

$$\pm m_p(z, \sigma), \qquad \pm m_s(z, \sigma)$$

Here
$$m_p^2(z,\sigma) = 1 - n_p^2(z)\sigma^2, \quad m_s^2(z,\sigma) = 1 - n_s^2(z)\sigma^2$$
$$n_p^2(z) = \frac{\rho}{\nu} = \frac{1}{v_p^2(z)}, \quad n_s^2(z) = \frac{\rho}{\mu} = \frac{1}{v_s^2(z)}$$

Evidently, for $\sigma < \min v_s(z)$ all the characteristic numbers are different (since $n_s > n_p$) and the asymptotic of Ia. D. Tamarkin used in [4] may be applied. For simplicity, let us consider that, starting with some depth $z_1 > 0$, the functions λ, μ, ρ are constant (*) The result of applying the classical asymptotic is: if the functions $\lambda(z), \mu(z), \rho(z)$ have three continuous derivatives, then the dispersion equation (1.11) has the asymptotic representation
$$\Delta_0(\sigma) + k^{-1}\Delta_1(\sigma) + O(k^{-2}) = 0 \tag{2.1}$$
$$\Delta_0(\sigma) \equiv |1 + m_s^2(0,\sigma)|^2 - 4m_p(0,\sigma)m_s(0,\sigma)$$
in the interval $0 < \varepsilon \leqslant \sigma \leqslant \min v_s(z) - \varepsilon$

Here $\Delta_0(\sigma)$ is the Rayleigh determinant for the homogeneous half-space $z \geq 0$. It follows from (2.1) that if the condition
$$v_{sm} \equiv \min v_s(z) > v_R \tag{2.2}$$
is satisfied (Fig. 1a), then for at least $k \gg 1$ there exists a solution of (2.1) of the form
$$\sigma_R(k) = v_R + k^{-1}v_1 + O(k^{-2}) \tag{2.3}$$

The coefficient v_1 has been found in [3].

In case the condition (2.2) is violated (Fig. 1b), the Tamarkin asymptotic is not applicable since there are turning points (we designate those values of z for which the characteristic numbers of the matrix H agree, as turning points).

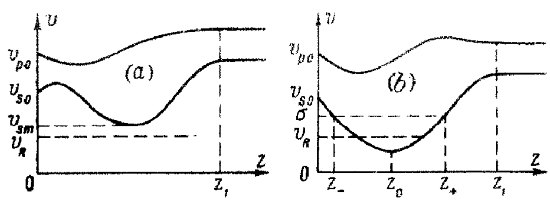


Fig. 1

The turning points of the system (1.8) have been encountered earlier in [1], where the requisite asymptotic has been constructed by a standard method. In the case under consideration, at least two turning points are encountered in the interval $z \geq 0$, which complicates the standard method. An asymptotic of the solutions of (1.8) could be found by joining the standard asymptotic; however, difficulties arise here in the investigation of the spectrum near the quasi-intersections. It is simplest to utilize the splitting method [6 and 7], although it imposes more stringent requirements on λ, μ, ρ than the standard method.

Let $\lambda(z), \mu(z), \rho(z)$ be infinitely differentiable, and starting with $z = z_1$, are constant. Let us examine the domain $0 < \sigma < \min v_p(z)$. The characteristic numbers of the matrix H decompose into three groups
$$(1) \quad -m_p(z,\sigma), \qquad (2) \quad +m_p(z,\sigma), \qquad (3) \quad \mp m_s(z,\sigma)$$
such that the numbers of one group do not equal the numbers of the other groups. Under

*) The Tamarkin formulas have been proved in the general case for a finite interval, however it may be shown [5] that they are valid for (1.8) even in the interval $0 \leq z < +\infty$ if λ, μ, ρ satisfy some conditions at infinity.

these conditions, the results of Feshchenko [6] and of Iliukhin [7] may be utilized, from which it follows that there exists a nondegenerate transformation

$$\mathbf{Z} = U(z, k, \sigma)\,\mathbf{X} \tag{2.4}$$

which reduces the system to quasi-diagonal form (2.5)

$$\mathbf{X}' = kB(z, k, \sigma)\,\mathbf{X}, \qquad B(z, k, \sigma) = [B_1(z, k, \sigma),\ B_2(z, k, \sigma),\ B_3(z, k, \sigma)]$$

$$B_1(z, k, \sigma) = -m_p(z, \sigma) - k^{-1}\frac{1}{2}\frac{(\rho m_p)'}{\rho m_p} + O(k^{-2})$$

$$B_2(z, k, \sigma) = m_p(z, \sigma) - k^{-1}\frac{1}{2}\frac{(\rho m_p)'}{\rho m_p} + O(k^{-2})$$

$$B_3(z, k, \sigma) = \begin{pmatrix} 0 & 1 \\ m_s^2(z, \sigma) & 0 \end{pmatrix} + k^{-1}\begin{pmatrix} a(z, \sigma) & 0 \\ 0 & b(z, \sigma) \end{pmatrix} + O(k^{-2})$$

$$a(z, \sigma) + b(z, \sigma) = -\rho'/\rho$$

Here

$$U(z, k, \sigma) = U_0(z, \sigma) + O(k^{-1})$$

$$U_0^{-1}(z, \sigma)\,H(z, \sigma)\,U_0(z, \sigma) = \left[\begin{pmatrix} -m_p(z, \sigma) & 0 \\ 0 & m_p(z, \sigma) \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ m_s^2(z, \sigma) & 0 \end{pmatrix}\right]$$

See the matrix U_0, say, in [1]. The fundamental matrix of the system (2.5) is also quasi-diagonal

$$X(z, k, \sigma) = [X_1(z, k, \sigma),\ X_2(z, k, \sigma),\ X_3(z, k, \sigma)] \tag{2.6}$$

Let us introduce the notation

$$\exp\left(k\int_x^y m_l(\zeta, \sigma)\,d\zeta\right) \equiv \varkappa_l(x, y) \qquad \left[\frac{m_l(0, \sigma)}{m_l(z, \sigma)}\right]^{1/2} \equiv \chi_l(z, \sigma) \qquad (l = p, s)$$

The asymptotic Formulas for X_1 and X_2 are obtained at once

$$X_1(z, k, \sigma) = \chi_p(z, \sigma)\,[\rho(0)/\rho(z)]^{1/2}\,\varkappa_p(z, 0)\,[1 + O(k^{-1})]$$

$$X_2(z, k, \sigma) = \chi_p(z, \sigma)\,[\rho(0)/\rho(z)]^{1/2}\,\varkappa_p(0, z)\,[1 + O(k^{-1})] \tag{2.7}$$

The system for X_3 may be reduced to one second order equation of the form

$$y'' - k^2 m_s^2(z, \sigma)y + O(1)y = 0 \tag{2.8}$$

Let us consider the function $v_s(z)$ to be monotonous for $z < z_0$ and for $z_0 < z < z_1$ where

$$\operatorname{sgn} v_s'(z) = \operatorname{sgn}(z - z_0) \qquad \text{for } 0 \leqslant z < z_1 \tag{2.9}$$

(Let us recall that the velocities v_s and v_p are constant for $z > z_1$.)
Henceforth, only the domain

$$0 < \sigma < v_M \qquad (v_M \equiv \min\{v_{s0};\ v_{s\infty};\ \min v_p(z)\},\ v_{s0} \equiv v_s(0),\ v_{s\infty} \equiv v_s(\infty))$$

is considered throughout.

Evidently if $\sigma < v_{sm} \equiv \min v_s(z)$, there are no turning points, and we have the case examined in [3]. For $v_{sm} < \sigma < v_M$ there are just two (and, moreover, simple because of (2.9)) turning points $z_-(\sigma)$ and $z_+(\sigma)$; at these points $m_s^2 = 0$.

We find the asymptotic of the solution of (2.8) (and thereby the asymptotic of the matrix X_3) in the presence of two turning points by merging. Namely, let $X_-(z, k, \sigma)$

and $X_{+}(z, k, \sigma)$ be some fundamental matrices of the system

$$X_3' = kB_3(z, k, \sigma) X_3$$

defined, respectively, for $z \leq z_0$ and $z \geq z_0$.

Evidently the matrix

$$X_3(z, k, \sigma) = \begin{cases} X_+(z, k, \sigma) & (z \geqslant z_0) \\ X_-(z, k, \sigma)\, X_-^{-1}(z_0, k, \sigma)\, X_+(z_0, k, \sigma) & (z \leqslant z_0) \end{cases} \tag{2.10}$$

may be a continuous extension of $X_{+}(z, k, \sigma)$ to the left from z_0.

It is convenient to take matrices having the asymptotic (on the appropriate intervals)

$$X_{\pm}(z, k, \sigma) = (\rho(z_0)/\rho(z))^{1/2} (E + O(k^{-1}))\, W_{\pm}(z, k, \sigma) \tag{2.11}$$

where

$$W_{\pm}(z, k, \sigma) = \begin{pmatrix} y_{1\pm} & y_{2\pm} \\ k^{-1}y_{1\pm}' & k^{-1}y_{2\pm}' \end{pmatrix}, \qquad \begin{aligned} y_{1\pm} &= 2k^{1/6}(\varphi_{\pm}')^{-1/2}\, v(k^{2/3}\varphi_{\pm}) \\ y_{2\pm} &= k^{1/6}(\varphi_{\pm}')^{-1/2}\, u(k^{2/3}\varphi_{\pm}) \end{aligned}$$

$$\varphi_{\pm} = \pm \left({}^3/_2 \int\limits_{z_{\pm}(\sigma)}^{z} |m_s(\zeta, \sigma)|\, d\zeta \right)^{2/3} \operatorname{sgn}(z - z_{\pm}(\sigma))$$

u and v are the Airy functions as defined by Fok [8], for X_+ and X_-.

Utilizing the Debye asymptotic of the Airy functions for $v_{sm} + \delta \leqslant \sigma \leqslant v_M - \delta$ we obtain

for $z \geqslant z_+(\sigma) + \varepsilon$

$$X_3(z, k, \sigma) = \left(\frac{\rho(z_0)}{\rho(z)}\right)^{1/2} \left\{ \begin{pmatrix} \gamma^{-1} & \gamma^{-1} \\ -\gamma & \gamma \end{pmatrix} + O_1 \right\} [\varkappa_s(z, z_+), \varkappa_s(z_+, z)] \tag{2.12}$$

for $0 \leqslant z \leqslant z_-(\sigma) - \varepsilon$

$$X_3(z, k, \sigma) = \left(\frac{\rho(z_0)}{\rho(z)}\right)^{1/2} \left\{ \begin{pmatrix} \gamma^{-1} & \gamma^{-1} \\ \gamma & -\gamma \end{pmatrix} + O_1 \right\} [\varkappa_s(z_-, z), \varkappa_s(z, z_-)]\, C \tag{2.13}$$

Here, and henceforth, the symbol O_1 replaces the symbol $O(k^{-1})$ and we have introduced the notation

$$C = C(k, \sigma) = \begin{pmatrix} -\cos\Phi + O_1 & {}^1/_2 \sin\Phi + O_1 \\ 2\sin\Phi + O_1 & \cos\Phi + O_1 \end{pmatrix}$$

$$\Phi = \Phi(k, \sigma) = \frac{\pi}{2} + k \int\limits_{z_-(\sigma)}^{z_+(\sigma)} |m_s(\zeta, \sigma)|\, d\zeta > 0, \qquad \gamma = \sqrt{m_s(z, \sigma)}$$

The matrix X_3 oscillates in the interval (z_-, z_+).

3. Dispersion curves. We shall designate graphs of the solution $\sigma = \sigma(k)$ of the dispersion Eq. (1.11) as the dispersion curves.

From (2.12) and (2.13), we obtain an asymptotic representation of the dispersion equation in the domain $v_{sm} + \delta \leqslant \sigma \leqslant v_M - \delta$

$$R(k, \sigma)\, R^*(k, \sigma) = {}^1/_2 S(k, \sigma)\, S^*(k, \sigma)\, e^{-2kf(\sigma)} \tag{3.1}$$

where

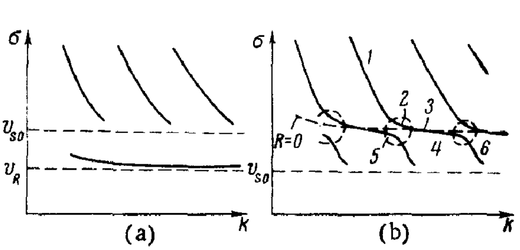
$$R(k, \sigma) = (1 + m_{s0}^2)^2 - 4m_{p0}m_{s0} + O_1, \quad m_{s0} = m_s(0, \sigma), \quad m_{p0} = m_p(0, \sigma)$$

$$R^{*}(k, \sigma) = \sin \Phi(k, \sigma) + O_1,$$
$$S(k, \sigma) = (1 + m_{s0}^2)^2 + 4m_{p0}m_{s0} + O_1, \qquad f(\sigma) = \int_0^{z_-(\sigma)} m_s(\zeta, \sigma)\, d\zeta > 0$$
$$S^{*}(k, \sigma) = \cos \Phi(k, \sigma) + O_1,$$

This is an equation of the same type as has been studied earlier in [1].

If the medium is such that $v_R < v_{sm}$ (case A), then there exists an "intrinsically Rayleigh" solution of Eq. (1.11) of type (2.3), and Eq. (3.1) yields a family of "waveguide" curves of simple form (Fig. 2a).

If, however, $v_{sm} < v_R < v_M$ (case B), then the picture is complicated (Fig. 2b). Characteristic here is the presence of points of an exponential approach of adjacent curves. The corresponding domains, encircled by dashes in Fig. 2b, we shall call domains (neighborhoods) of quasi-intersection.



(a) (b)

Fig. 2

Let us divide each dispersion curve in case B into sections 1, ..., 6, as is shown in Fig. 2b. The sections 2 and 5 lie in the quasi-intersection neighborhoods; the line $R(k, \sigma) = 0$ (dashed) is intersected by the dispersion curves on the boundary of the sections 3 and 4.

In the next Section, the behavior of the solution of the problem (1.5) to (1.7) is considered for $k \gg 1$ as a function of z for $0 < \delta \leqslant \sigma \leqslant v_{sm} - \delta$ and for $v_{sm} + \delta \leqslant \sigma \leqslant \leqslant v_M - \delta$. For simplicity, the first component G_1 of the eigenfunctions in the intervals $0 \leqslant z \leqslant z_- - \varepsilon$, $z_- + \varepsilon \leqslant z \leqslant z_+ - \varepsilon$ and $z \geqslant z_+ + \varepsilon$, will be investigated, which will correspond to replacement of the Airy functions by their Debye asymptotic, although our formulas of Section 2 permit also the investigation of the domain $\sigma \approx v_{sm}$ and the behavior of the eigenfunctions in the whole $z \geq 0$ interval including the neighborhoods of the points z_- and z_+.

4. Dependence of the solutions on z. Let us proceed from the asymptotic formulas of Section 2 and the expressions for the eigenfunctions

$$G(z, k, \sigma) = m\,[D_p(k, \sigma) G^s(z, k, \sigma) - D_s(k, \sigma) G^p(z, k, \sigma)]$$

where m is an arbitrary factor independent of z (we henceforth omit this factor).

In case A the intrinsic Rayleigh oscillation is described for $z \geq 0$ by

$$G_1(z, k, \sigma) = O(\varkappa_s(z, 0)) \tag{4.1}$$

For $v_{sm} + \delta \leqslant \sigma \leqslant v_M - \delta$ we have in both cases in the interval $0 \leq z \leq z(\sigma) - \epsilon$:

$$G_1(z, k, \sigma) = S\varkappa_s(z, 0) - (1 + O_1) R\varkappa_s(0, z) + \tau\varkappa_p(z, 0) \tag{4.2}$$

$$\tau = \tau(z, k, \sigma) = 4(1 + m_{s0}^2 + O_1)\chi_p \chi_s > 0$$

The functions $\varkappa_l$, χ_l ($l = p, s$) are defined in Section 2.

In case A for the waveguide oscillations $R \geq \epsilon > 0$.

For $z = 0$

$$G_1 = S - R + \tau + O_1 > 0$$

For $z \geq \epsilon > 0$, $k \gg 1$ the function $G_1 < 0$, therefore, near $z = 0$ there is a zero $z^{(0)}(k)$ of the function $G_1(z, k, \sigma)$. It is easy to see that

$$z^{(0)}(k) = O_1 > 0 \tag{4.3}$$

At the point $z^{(2)}(k) = z^{(0)}(k) + O_1$ the graph of $G_1(z)$ has an inflection; the derivative $dG_1/dz < 0$ for $0 \leq z \leq z_- - \epsilon$.

Thus for waveguide oscillations

$$G_1(z, k, \sigma) = O(\varkappa_s(0, z)) \qquad (z^{(0)} \leqslant z \leqslant z_- - \varepsilon) \tag{4.4}$$

The waveguide-type eigenfunctions in the interval $z_- + \varepsilon \leqslant z \leqslant z_+ - \varepsilon$ oscillate in conformity with Formula

$$G_1(z, k, \sigma) = \left(\frac{|m_s(z, \sigma)|}{\rho(z)}\right)^{1/2} (\cos\Psi + O_1) \tag{4.5}$$

$$\Psi = \Psi(z, k, \sigma) = k \int\limits_{z_-(\sigma)}^{z} |m_s(\zeta, \sigma)|\, d\zeta + \frac{\pi}{4}$$

$$G_1(z, k, \sigma) = O(\varkappa_s(z, z_+)) \qquad \text{for } z \geqslant z_+ + \varepsilon \tag{4.6}$$

Graphs of $G_1(z)$ corresponding to case A are pictured schematically in Fig. 3 (the normalization of the eigenfunctions is defined by considerations of convenience for the sketch).

Let us consider case B. For $k \gg 1$ the derivative $\partial R/\partial\sigma > 0$, hence the line $R(k, \sigma) = 0$ (Fig. 2b, dashed) divides each dispersion curve into

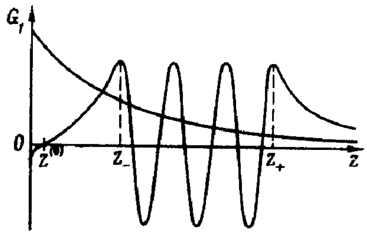


Fig. 3

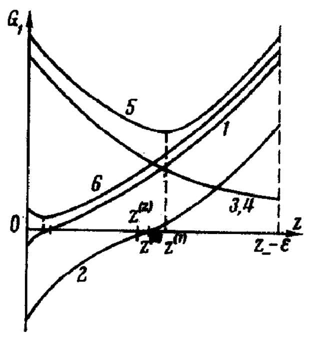


Fig. 4

two parts: $R > 0$ on sections 1, 2, 3, and $R < 0$ on sections 4, 5, 6. For section 1 we have exactly the same as for the waveguide oscillations in case A. For section 3

$$|R^*| \sim 1, \qquad R = SS^*(2R^*)^{-1} \exp(-2kf(\sigma)) > 0$$

Hence, for $0 \leqslant z \leqslant z_- - \varepsilon$

$$G_1(z, k, \sigma) = S\varkappa_s(z, 0) - SS^*/2R^*\varkappa_s^2(z_-, 0)\varkappa_s(0, z) + \tau\varkappa_p(z, 0) \tag{4.7}$$

It is seen that for sufficiently large k the function $G_1(z)$ is monotonously decreasing in the interval $(0, z_- - \epsilon)$.

As a point moves along the dispersion curve from section 1 to section 3, the quantity R, remaining positive, changes order from 1 to $\exp(-2kf)$. It is easy to see that this causes the motion of the zero $z^{(1)}(k)$ of the function $G_1(z)$ from $z = O_1$ to $z = z_- - \epsilon$.

On section 4 of type $R < 0$, however, $G_1(z)$ is monotonous as before in the interval $(0, z_- - \epsilon)$. On section 5, $G_1(z)$ has an extremum at the point $z^{(1)}(k)$, which is displaced from $z = O_1$ to $z = z_- - \epsilon$ as the point (k, δ) moves along the dispersion curve from the section 6 to section 4.

Now $_{\cdot}^{(1)}(k) = O_1$ on section 6. Graphs of the dependence $G_1(z)$ in the interval $[0, z_- - \epsilon]$ are given schematically for case B in Fig. 4.

The eigenfunctions oscillate in the interval $z_- + \varepsilon \leqslant z \leqslant z_+ - \varepsilon$, and for $z \geq z_+ + \epsilon$ decrease monotonously as in case A.

Therefore, in case A $(v_R < \min v_s(z))$ the intrinsic Rayleigh wave has completely usual properties since its amplitude decreases as $\exp(-kzc)$ $(c > 0)$ for all z. In case B $(v_R > \min v_s(z))$ waves close to the customary Rayleigh waves in their properties will correspond to the intrinsically Rayleigh portions (3 and 4) of the dispersion curves; they are noticeable only near the surface, and have a small oscillation within the waveguide. In both case A and case B the intrinsically waveguide waves possess the customary properties; they are noticeable only within the waveguide. In case B there are also solutions of transition type, corresponding to quasi-intersection neighborhoods (sections 2 and 5). These waves are noticeable both near the surface, and also within the waveguide. Apparently these waves are of interference character.

In conclusion, let us note that the expounded method is applicable without substantial changes, even in the case when λ, μ, ρ (or their derivatives) have jumps at $z = z_0$, and are infinitely smooth otherwise.

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